Tips for Quick Solution:-

(i) Only in first iteration, since Δ_j's are the same as $-C_j$'s so there is no need to calculating them separately by using the formula $\Delta_j = C_B X_j - C_j$.

(ii) Mark min. (Δ_j) by ↑ which at once indicates the column x_k needed for computing the min. ratio X_B/X_k.

(iii) 'Key element' is found at the place where the upward direction arrow (↑) of min. of Δ_j and the right directed arrow (⟶) of minimum ratio (X_B/X_k) intersect each other in the simplex table.

(iv) Key element indicates that the current table must be transformed in such a way that the key element becomes 1 and all other elements in that column become zero.

(v) Since Δ_j's corresponding to unit column vector are always zero, there is no need of calculating them.
While transforming the table by row operation the value of Z and corresponding Δ_j's are also computed at the same time.
Thus a lot of time and labour can be saved in adopting this technique.

Simple way for simplex Method Computations:

Example: Consider the LP problem

$$\text{Max. } z = 3x_1 + 2x_2$$

$$\text{s.t.o } x_1 + x_2 \le 4$$

$$x_1 - x_2 \le 2$$

$$\text{and } x_1, x_2 \ge 0$$

Solution: The complete solution with it's different computational steps can be more conveniently represented by following table:

Basic Variables	C_B	X_B	x_1 (C_j = 3)	x_2 (C_j = 2)	s_1 (C_j = 0)	s_2 (C_j = 0)	Min. Ratio
s_1	0	4	1	1	1	0	4/1
s_2	0	2	1	-1	0	1	2/1 Min.
		$z = C_B X_B = 0$	$-3\uparrow$	-2	0	$0\downarrow$	
s_1	0	2	0	+2	1	-1	2/2 min →
x_1	3	2	1	-1	0	1	− (-ve)
		$z = C_B X_B = 6$	0	$-5\uparrow$	$0\downarrow$	3	
x_2	2	1	0	1	1/2	-1/2	
x_1	3	3	1	0	1/2	1/2	
		$z = C_B X_B = 11$	0	0	5/2	1/2	

Thus the optimal solution is obtained as

$$x_1 = 3, \quad x_2 = 1 \quad \text{and Max. } z = 11$$

Example: Solve the LPP by simplex method

Max. $z = 2x_1 + 5x_2$ subject to the constraints

$x_1 + 3x_2 \le 3$, $3x_1 + 2x_2 \le 6$, $x_1, x_2 \ge 0$.

Solution: In this problem, introducing slack variables s_1 and s_2, the problem becomes

Max. $z = 2x_1 + 5x_2 + 0 \cdot s_1 + 0 \cdot s_2$ subject to the constraints

$$x_1 + 3x_2 + s_1 = 3$$
$$3x_1 + 2x_2 + s_2 = 6$$
and $x_1, x_2 \ge 0$

Taking $x_1 = x_2 = 0$, we get $s_1 = 3$, $s_2 = 6$ which is starting B.F.S. All computational work is done in following table:

		$C_j \rightarrow$	2	5	0	0	
Basic variables	C_B	x_B	x_1	x_2	s_1	s_2	Min. Ratio
s_1	0	3	1	3	1	0	3/3 ← Min
s_2	0	6	3	2	0	1	6/2
$x_1 = x_2 = 0$	$z = 0$	-	-2	-5	0	0	← Δ_j
x_2	5	1	1/3	1	1/3	0	3/1
s_2	0	4	7/3	0	-2/3	1	12/7 ← Min.
$x_1 = s_1 = 0$	$z = 5$		-1/3	0	5/3	0	← Δ_j
x_2	5	3/7	0	1	3/7	1/7	
x_1	2	12/7	1	0	-2/7	3/7	
$s_1 = s_2 = 0$	$z = 39/7$		0	0	11/7	1/7	

Since all $\Delta_j \ge 0$, the solution given by $x_1 = 12/7$, $x_2 = 3/7$ and Max. $z = 39/7$ is optimal.

<u>Example:-</u> Solve L.P.P. by simplex method.

$$\max. z = 3x_1 + 5x_2 \text{ subject to constraints}$$

$$3x_1 + 2x_2 \le 10, \quad x_1 \le 4$$

$$x_2 \le 4, \quad x_1, x_2 \ge 0$$

<u>Solution:-</u> In this problem introducing slack variables s_1, s_2 and s_3 the problem becomes $\max z = 3x_1 + 5x_2 + 0 \cdot s_1 + 0 s_2 + 0 s_3$ subject to the constraints

$$3x_1 + 2x_2 + s_1 = 18, \quad x_1 + s_2 = 4$$

$$x_2 + s_3 = 4 \quad \text{and } x_1, x_2 \ge 0$$

Taking $x_1 = x_2 = 0$, we get $s_1 = 18$, $s_2 = 4$ and $s_3 = 4$ which is starting B.F.S. All computational work is done in following table:

	$C_j \rightarrow$		3	5	0	0	0	
Basic variables	C_B	X_B	x_1	x_2	s_1	s_2	s_3	Min Ratio (X_B/X_k)
s_1	0	18	3	2	1	0	0	18/2
s_2	0	4	0		0	1	0	4/0
s_3	0	4	0	1	0	0	1	4/1 ← Min.
$x_1 = x_2 = 0$	$z = 0$		−3	−5	0	0	0	← Δ_j
s_1	0	10	3	0 ↑	1	0	−2	10/3 ← Min.
s_2	0	4	1	0	0	1	0	4/1
x_2	5	4	3	1	0	0	1	4/0
$x_1 = s_3 = 0$	$z = 20$		−3	0	0	0	5	← Δ_j
x_1	3	10/3	1 ↑	0	1/3	0	−2/3	
s_2	0	2/3	0	0	−1/3	1	2/3	
x_2	5	4	0	1	0	0	1	
$s_1 = s_3 = 0$	$z = 30$		0	0	1	0	3	← Δ_j

Since all $\Delta_j \ge 0$, the solution given by

$x_1 = 10/3$, $x_2 = 4$, $\max. z = 30$ is optimal.

EXPT NO ....

Artificial Variable Techniques:

Two Phase Method:-

Linear programming problem in which constraints may also have '≥' and '=' signs after ensuring that all b_i's are ≥ 0 are considered in this section. In such problems we cannot obtain basis matrix as an identity matrix in starting simplex table. Therefore we introduce a new type of variable called the 'artificial variable'. These variables are fictitious and cannot have any physical meaning.

The artificial variable technique is merely device to get the starting B.F.S so that simplex procedure may be adopted as usual until the optimal solution is obtained.

Artificial variables can be eliminated from the simplex table as and when they become zero (non basic). The process of eliminating artificial variables is performed in phase I of the solution. Phase II is used to obtain an optimal solution. Since, the solution of the LP problem is completed in two phase, it is called 'Two Phase Simplex method.'

Procedure of Two Phase Method:

(i) Express the LP problem in standard form.

(ii) Introduce artificial variables in all constraints of '≥' and '=' type.

(iii) Formulate a new objective function Z^* by assigning a cost -1 to each artificial variable and a cost zero to all other variables i.e.

$$\text{Max } Z^* = -A_1 - A_2 - \cdots - A_m$$

Now applying the simplex method to the LP problem with above objective function and constraints given in original LP problem.

In final (optimal) table of phase I there may be two cases:

Case I: Max $Z^* < 0$

This implies that at least one artificial variable is present in the basis at positive level. In this case LP problem has no F.o solution.

Case II: Max $Z^* = 0$

In this case there may be two situations. either there is no artificial variable in the optimum basis or all the artificial variables present in the final table are at zero level.

If there is case II, we go to phase II.

Phase II: In the final table, obtained at the end of phase I, we assigns the actual costs to the variables and zero cost to each artificial variables present in the basis at 0 level.

Remark:- (i) The objective of phase I is to search for a B.F.S. to the given problem it ends up either giving a B.F.S. or indicating that the given L.P.P. has no feasible solution at all.

(ii) The B.F.S. obtained at the end of phase I provides a starting B.F.S. for the given L.P.P. Phase II is then just the application of simplex method to move towards optimality.

(iii) In phase II are must be taken to ensure that an artificial variable is never allowed to become positive if were present in the basis. Moreover whenever some artificial variables happens to leave the basis, it column must be deleted from the simplex table altogether.

Example: Solve the problem

$$\text{Min. } z = x_1 + x_2$$

s.to $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

Solution: First of all convert the given minimization problem to maximization problem by writing the objective function as

$$\text{Max. } (-z) = -x_1 - x_2$$

$$\text{Max. } z^* = -x_1 - x_2, \text{ where } z^* = -z$$

$\because$ all b_i's $(4, 7)$ are +ve the 'surplus variables' $S_1 \geq 0$ and $S_2 \geq 0$ are introduced, the constraints becomes

$$2x_1 + x_2 - S_1 = 4, \quad x_1 + 7x_2 - S_2 = 7$$

Here the basis matrix B would not be an identity matrix. Hence starting B.F.S. cannot be obtained.

Therefore to get identity matrix a new type of variable so called artificial variables $A_1 \geq$ and $A_2 \geq 0$ are introduced as:

$$2x_1 + x_2 - S_1 + A_1 = 4$$

$$x_1 + 7x_2 - S_2 + A_2 = 7$$

Phase I: Auxiliary objective function takes the form

$$\text{Max. } z^{**} = 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 + (-1) A_1 + (-1) A_2$$

$$= 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot S_1 + 0 \cdot S_2 - 1 \cdot A_1 - 1 \cdot A_2$$

Table:

Basic variables	C_B	X_B	x_1 (0)	x_2 (0)	s_1 (0)	s_2 (0)	A_1 (-1)	A_2 (-1)	Min Ratio
A_1	-1	4	2	1	-1	0	1	0	4/1
A_2	-1	7	1	7	0	-1	0	1	7/7 →
	$z^{**} = -11$		-3	-8 ↑	1	1	0	0 ↓	
A_1	-1	3	13/7	0	-1	1/7	1		21/13 →
x_2	0	1	1/7	1	0	-1/7	0		7/1
	$z^{**} = -3$		-13/7 ↑	0	1	-1/7	0		
x_1	0	21/13	1	0	-7/13	1/13			
x_2	0	10/13	0	1	1/13	-2/13			

Phase II : Now in order to test the starting above solution for optimality construct the starting simplex table as:

Basic variables	C_B	X_B	x_1	x_2	s_1	s_2
x_1	-1	21/13	1	0	-7/13	1/13
x_2	-1	10/13	0	1	1/13	-2/13
		$z^* = -31/13$	0	0	6/13	1/13

Optimal solution is $x_1 = 21/(13) = 21/3 = 7$

$x_2 = 10/(3) = 10/3 = 3\frac{1}{3}$

and $z^* = -31/3$ or Min $z = 31/3 = 10\frac{1}{3}$.

Big-M- Method (Charne's Penalty Method):-

Big-M method is an alternative approach of solving a linear programming problem involving artificial variables. In this method we assign a very high penalty (say M) to the artificial variable in the objective function.

Computational Step for Big-M- Method:

The computation steps for Big-M-Method are given below:

(i) Express the given problem in standard form.

(ii) Add non-negative artificial variables to the left side of each of the equation corresponding to constraints of the type (≥) and '='.

When artificial variables are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution (provided the solution of the problem exists.).

(III) If the problem does not have a solution at least one of artificial variable will appear in the final solution with positive value. This is achieved by assigning a very large price (per unit penalty) to these variables in the objective function. Such large price to each artificial variable is denoted by (-M) for maximization problem and $(+M)$ for minimization problems where $M > 0$.

(iv) At last artificial variables are used for starting solution and then usual simplex method is used to obtain optimal soln.
Such a method used to solve an LP problem is called penalty method or Big-M-Method.

Example:- Using Big-M method solve the following LP problem:

$$\text{Max. } Z = -2x_1 - x_2$$

$$\text{s.to } 3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0.$$

Solution:- Step (I):- Introducing slack, surplus & artificial variables, the system of constraint equation belows:

$$3x_1 + x_2 + a_1 = 3$$

$$4x_1 + 3x_2 - s_1 + a_2 = 6$$

$$x_1 + 2x_3 + s_2 = 4$$

Where s_1 is surplus, s_2 is slack and a_1, a_2 are artificial variable.

Step II:- Since, the given problem is of maximization, assigning the large value (-ve) $-M$ to the artificial variables a_1 and a_2 the objective function becomes:

Max. $Z = -2x_1 - x_2 + 0 \cdot S_1 + 0 \cdot S_2 - M \cdot a_1 - M \cdot a_2$

Step III: Now constructing the starting simplex table as following:

	$C_j \rightarrow$		-2	-1	0	0	$-M$	$-M$	
Basic variables	C_B	X_B	x_1	x_2	S_1	S_2	a_1	a_2	Min. Ratio
a_1	$-M$	3	3	1	0	0	1	0	$3/3 \rightarrow$
a_2	$-M$	6	4	3	-1	0	0	1	$6/4$
S_1	0	4	1	2	0	1	0	0	$4/2$
		$Z = -9M$	$\uparrow (2-7M)$	$(1-4M)$	M	0	$\downarrow 0$	0	

Using minimum Ratio Rule, find the key element 3 which indicates that a_1 should be removed. Now the transformed table is obtained by usual manner:

	$C_j \rightarrow$		-2	-1	0	0	$-M$	$-M$	
Basic variables	C_B	X_B	x_1	x_2	S_1	S_2	a_1	a_2	Mini. Ratio
x_1	-2	1	1	$1/3$	0	0	$1/3$	0	$1/\frac{1}{3} \rightarrow$
a_2	$-M$	2	0	$5/3$	1	0	$4/3$	1	$2/\frac{5}{3}$
S_2	0	3	0	$5/3$	0	1	$-1/3$	0	$3/\frac{5}{3}$
		$Z = -2-2M$	0	$\uparrow \left(\frac{1-5M}{3}\right)$	M	0	$\frac{-2-4M}{3}$	0	

Since minimum Δ_j rule and minimum ratio rule decide the key element is $5/3$, so incoming vector is x_2 and a_2 is outgoing. Therefore the second

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improved table is formed:

$C_j \rightarrow$

Basic variables	C_B	X_B	x_1	x_2	S_1	S_2	a_1	a_2	Min. Ratio
x_1	-2	3/5	1	0	1/5	0	3/5	-1/5	
x_2	-1	6/5	0	1	-3/5	0	-4/5	3/5	
S_2	0	1	0	0	1	1	1	-1	
		$z=-12/5$	0	0	1/5	0	$M-2/5$	$M-\frac{2}{5}$	

To test the solution of optimality compute:

$\Delta_3 = C_B X_3 - C_3 = (-2,-1,0)(1/5,-3/5,1) - 0 = 1/5$

$\Delta_5 = C_B a_2 - C_5 = (-2,-1,0)(-1/5,-3/5,-1) + M$

$= M - 2/5$

$\Delta_6 = C_B a_2 - C_6 = (-2,-1,0)(-1/5,-3/5,-1) + M$

$= M - 2/5$

since, M is as large as possible, $\Delta_3, \Delta_5, \Delta_6$ are all positive consequently the optimal solution is

$x_1 = 3/5, \quad x_2 = 6/5$

& Max. $z = -12/5$